Accuracy of Verdicts under Different Jury Sizes and Voting Rules

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Juries are a fundamental element of the criminal justice system. In this article, we model jury decision making as a function of two institutional variables: jury size and voting requirement. We expose the critical interdependence of these two elements in minimizing the probabilities of wrongful convictions, of wrongful acquittals, and of hung juries. We find that the use of either large nonunanimous juries or small unanimous juries offers alternative ways to maximize the accuracy of verdicts while preserving the functionality of juries. Our framework, which lends support to the elimination of the unanimity requirement in the presence of large juries, helps appraise US Supreme Court decisions and state legal reforms that have transformed the structure of American juries.
1. INTRODUCTION

Jury design is a critical element of criminal adjudication. After more than six centuries without change, the structure and functioning of juries have recently undergone several significant transformations regarding jury size and voting requirements. Juries in a criminal case were traditionally composed of 12 members, who needed to reach a unanimous agreement to render a decision.1 Although most Americans view the 12-member jury as a fixture of American legal procedure, several US Supreme Court decisions have affirmed the constitutionality of juries with fewer than 12 members, as well as juries operating under a voting requirement less stringent than unanimity. This article seeks to evaluate the desirability—or lack thereof—of these institutional transformations by analyzing the impact of changes in jury size and voting requirements on the probability of wrongful convictions, wrongful acquittals (i.e., convicting the innocent and acquitting the guilty, respectively), and hung juries.

Prior literature on jury design has investigated jury size and voting requirements as independent policy variables or in pairwise choice frameworks, but this literature has often neglected the critical interdependence of jury size and voting requirements in maximizing the accuracy of verdicts. Prior contributions have separately investigated how large a jury should be (Paroush 1997; Ben-Yashar and Paroush 2000; Dharmapala and McAdams 2003; Helland and Raviv 2008; Luppi and Parisi 2013) and how juries should vote to reach an accurate verdict (Klevorick and Rothschild 1979; Klevorick, Rothschild, and Winship 1984; Ladha 1995; Young 1995; Neilson and Winter 2005).

Our article contributes to the existing literature by exposing the critical interplay between jury size and voting requirement in criminal adjudication. We extend the criminal trial model developed in Neilson and Winter (2000, 2005) by both relaxing the unanimity requirement and varying the jury size. We investigate how different combinations of these two institutional variables affect the probabilities of accurate verdicts, wrongful verdicts, and hung juries. Our results reveal that jury size and voting requirements should inversely depend on one another: large nonunanimous juries or small unanimous juries are alternative solutions to maximize the accuracy of verdicts. We discuss these findings in the light of recent legal transformations to jury structure, and we offer insights for policy analysis.

The article is organized as follows. Section 2 briefly reviews the legal and economic backgrounds on jury design. Section 3 presents

1 In the leading 1898 case Thompson v. Utah, the Court construed the Sixth Amendment to require that in all criminal cases, a jury must be composed of exactly 12 persons.
the criminal trial model. Section 4 introduces a numerical example to investigate how different combinations of a jury’s institutional characteristics affect the probability of wrongful convictions and wrongful acquittals, as well as the ability of the jury to reach a deliberation. Section 5 concludes with a discussion of our results and their relevance for policy purposes.

2. RELATED LITERATURE

For the last six centuries, criminal verdicts have been rendered by juries composed of 12 members, deliberating unanimously. In recent years, the US Supreme Court has granted states the freedom to reduce the size of juries and to relax the jury’s voting requirement, allowing nonunanimous verdicts. The changes have taken place through a series of cases decided by the US Supreme Court between 1968 and 1979. In one of these cases, the well-known Williams v. Florida, the Supreme Court recognized that a verdict rendered unanimously by fewer than 12 jurors was not inconsistent with the constitutional right to have a trial by jury. In a subsequent decision, Ballew v. Georgia, the Supreme Court set a lower limit on jury size, affirming that any jury with fewer than six members would be unconstitutional because it would be too small to be representative of the relevant community.

Other important changes took place with respect to the jury’s voting requirement. Unanimity for criminal verdicts has generally been viewed as an important requirement to preserve the public confidence in the criminal justice system, because wrongful convictions of innocent defendants are less likely under unanimity (Coughlan 2000). However, unanimity allows any single juror to veto a proposed verdict and single-handedly lead to a mistrial. The increasing administrative and financial cost of mistrials led some states to consider criminal justice reforms that relaxed the unanimity requirement. These state reforms were challenged at the federal level.

In the leading cases—namely, Duncan v. Louisiana, Johnson v. Louisiana, and Apodaca v. Oregon—the US Supreme Court ruled that verdicts reached under a qualified majority rule do not violate the US Constitution.

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5 See, e.g., Hannaford-Agor et al.’s (2002) NCSC (National Center for State Courts) multiphased research on mistrials, motivated by the concern that mistrials were reaching unacceptably high levels in some jurisdictions. See also Kalven and Zeisel’s (1966) study of the American jury, which briefly discussed the phenomenon of mistrials in criminal cases.
Constitution. This ruling gave states the flexibility to pursue criminal justice reforms by allowing verdicts to be reached under a qualified majority rule. In 1979, the Court in *Burch v. Louisiana* held that states could either reduce jury size or lessen the voting requirement but not both simultaneously: nonunanimous verdicts could only be rendered by juries of 12, and smaller juries could only deliberate unanimously. As of today, only Oklahoma, Oregon, and Louisiana allow nonunanimous verdicts in misdemeanor cases; Oregon and Louisiana also allow them in felony cases.

The abolition of the unanimity requirement for criminal verdicts was met with a mixture of approval and skepticism. Supporters viewed nonunanimous decision-making as a possible solution to the hung-jury problem (e.g., Amar 1994; Glasser 1996; Morehead 1997). Opponents viewed the abolition of the unanimity requirement as a violation of a fundamental principle of criminal justice for the protection of innocent defendants (e.g., Kachmar 1996; Smith 1996; Klein and Klastorin 1999). The views in the literature are split, revealing an objective difficulty in balancing the policy goals of accuracy in adjudication and reduction of the costs of criminal justice.

Several law and economics contributions have investigated the effects of changing jury size on the expected trial outcomes. A central argument in the literature on juries and jury decision making is that a group will make a better decision than an individual (Condorcet’s jury theorem). Some contributions refined Condorcet’s jury theorem and demonstrated that, under certain conditions, this theorem does not hold (e.g., in the presence of strategic voting, as shown by Feddersen and Pesendorfer 1998). For example, larger, unanimous juries may be more likely to reach an accurate verdict but may fail to reach any decision at all. Hence, a trade-off emerges between accuracy and

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8 See Oregon Revised Statutes §136.450, and Louisiana Laws Code of Criminal Procedure 782. Several states permit nonunanimous verdicts in civil trials. See State Court Organization, 1998, Figure 42 [Trial Juries: Size and Verdict Rules]. For more extensive discussions on these state regulations and mistrials, see Hannaford-Agor et al. [2002] and Luppi and Parisi [2013].
9 See also Ben-Yashar and Nitzan [1997], proving that the optimal rule for fixed-size committees in dichotomous choice situations is the qualified weighted majority. Feddersen and Pesendorfer [1998] showed that when jurors behave strategically, the probability of convicting the innocent in large juries is higher under the unanimity rule than under qualified majority rules. When there is uncertainty about jurors’ preferences, in the presence of strategic jurors with private information, the unanimity rule may still be preferable to protect innocent defendants against wrongful convictions [Luppi and Parisi 2013].
decisiveness [Luppi and Parisi 2013]. Notwithstanding the widespread adoption of smaller juries in state criminal courts, statistics indicate that overall mistrial rates have not declined (Kalven and Zeisel 1966; Hannaford-Agor et al. 2002). A few empirical studies have attempted to evaluate how jury size affects trial results. Most of them concluded that there is no detectable difference between 6-member and 12-member juries with respect to mistrial rates (e.g., Hannaford-Agor et al. 2002; Eisenberg et al. 2005). By contrast, experimental studies and statistical models on jury size found that jury size does affect trial outcomes and jurors’ behavior (e.g., Guarnaschelli, McKelvey, and Palfrey 2000; Mukhopadhaya 2003; Helland and Raviv 2008). For example, Guarnaschelli, McKelvey, and Palfrey (2000) revealed that larger juries may convict fewer innocent defendants than smaller juries under unanimity.

Our key original contribution to the literature is the specification of a different objective function that should guide the design of juries. Whereas previous studies have focused on either jury size or voting requirement, in this article, we reveal the crucial interdependence of these two variables, and we analyze their optimal combination in minimizing the probabilities of wrongful convictions and hung juries.

3. CRIMINAL TRIAL MODEL

In this section, we construct a simple model of the criminal trial process to analyze how varying jury size and voting requirement affects different expected trial outcomes.10 We follow the classical jury model (e.g., Miceli 1990; Feddersen and Pesendorfer 1998; Coughlan 2000; Neilson and Winter 2000, 2005; Duggan and Martinelli 2001; Persico 2004). There are two states of the world, $I$ and $G$ (Innocent and Guilty). Let $P(G)$ denote the prior probability that the defendant is guilty and $1 - P(G)$ the prior probability that the defendant is innocent. Let $s$ be the strength of evidence found against the defendant, whereby stronger evidence is associated with a higher probability of guilt. Let $f(s|G)$ and $f(s|I)$ be the probability density functions of the strength of evidence given that the defendant is guilty or innocent, respectively, and let $F(s|G)$ and $F(s|I)$ be the corresponding cumulative functions. The two density functions are represented in figure 1.

10 Our model relies on Neilson and Winter’s (2005) theoretical setup with the main difference that we vary not only voting requirements but also jury size. For a similar formulation of the court’s problem, see also Rubinfeld and Sappington (1987).
The traditional standard of proof in criminal trials in the United States is proof beyond a reasonable doubt, where each juror must individually believe in the guilt of the accused beyond any reasonable doubt.\textsuperscript{11} As in Neilson and Winter (2000, 2005), to model the reasonable-doubt standard we assume that some evidence is inconsistent with an innocent defendant. Specifically, an innocent defendant can generate evidence in the interval $[0, s_I]$, whereas a guilty defendant can generate evidence in the interval $[s_G, 1)$, with $0 \leq s_G < s_I < 1$. If a juror observes $s \geq s_I$, that juror can state that the defendant is guilty beyond a reasonable doubt. The opposite holds when $s < s_I$. Put another way, $s_I$ represents the reasonable-doubt standard threshold.\textsuperscript{12}

\textsuperscript{11} The beyond-a-reasonable-doubt standard has been used in criminal trials since at least the 1700s. It was adopted by most jurisdictions even before the case \textit{In re Winship} [397 U.S. 358, 1970] and recognized as a constitutional requirement.

\textsuperscript{12} The reasonable-doubt standard threshold follows from Judge Blackstone’s dictum that it is “better that ten guilty persons escape than that one innocent suffer” (Blackstone 1769). Blackstone’s ratio of 10 to 1—or any variation of such ratio in state case law (Rizzolli and Saraceno 2013; Pi, Parisi, and Luppi 2020)—follows from the fact that a wrongful conviction in criminal adjudication (i.e., convicting the innocent) is perceived to be worse than a wrongful acquittal (i.e., acquitting the guilty). For a
Analytically, this is equivalent to assuming that the probability density function for a guilty defendant first-order stochastically dominates the probability distribution function of an innocent defendant. Thus, under first-order stochastic dominance, it is more likely to find incriminating evidence for a guilty defendant than an innocent defendant.\(^\text{13}\) Graphically, the first-order stochastic dominance is represented by the fact that the \(f(s|G)\) distribution is shifted further to the right than the \(f(s|I)\) distribution.\(^\text{14}\)

As in Neilson and Winter (2005), we introduce juror heterogeneity by assuming that jurors do not directly observe the true evidence \(s\), but they rather observe signals of varying strength related to the evidence.\(^\text{15}\) Juror heterogeneity is a necessary assumption: if all individual jurors were perfectly able to observe the true strength of evidence, juries would always reach unanimous verdicts. However, this is not the case in real-world criminal trials, as the actual rates of hung juries and judicial errors show (e.g., Hannaford-Agor et al. 2002).

Each juror assesses evidence differently and, as a result, can express different opinions when deliberating for a verdict. Another interpretation of the weak/strong signal is that jurors are heterogeneously informed. This may be driven by differing levels of juror sensitivity to the arguments presented by the prosecutor or defense counsel, or it may be driven by other factors that affect the persuasion of relevant facts or evidence presented at trial.

Specifically, each juror has a probability \(\pi \in [0, 1]\) of receiving a strong signal of incriminating evidence, \(s_s = s + x\), with \(x \geq 0\), and a probability \(1 - \pi\) of receiving a weak signal of incriminating evidence,
sw = s - y < ss, with y ≥ 0. A juror who receives the strong signal votes to convict if ss ≥ sI, that is, if s ≥ sI - x, as represented in figure 1. A juror who receives the weak signal votes to convict if sw ≥ sI, that is, if s ≥ sI + y, as represented in figure 1. In a nutshell, a juror receiving the strong signal is more likely to believe that the defendant is guilty beyond any reasonable doubt than is a juror receiving the weak signal.

Let N ∈ [1, 2, ..., n] denote the size of a jury, and m ∈ [0, 1] denote the majority to reach a verdict. For the majority rule case, mN is the smallest integer greater than N/2; for the unanimity case, m = 1.

Let X be a random variable that follows the binomial distribution $X \sim B[N, \pi]$. The probability that mN jurors receive the strong signal (or, equivalently, vote to convict) $P_C = \Pr[X = mN] = \binom{N}{mN} \pi^{mN}(1 - \pi)^{N-mN}$. Similarly, let Y be a random variable which follows the binomial distribution $Y \sim B[N, 1 - \pi]$. The probability that mN jurors receive the weak signal (or, equivalently, vote to acquit) is $P_A = \Pr[Y = mN] = \binom{N}{mN} (1 - \pi)^{mN} \pi^{N-mN}$. The probability that a jury is neither prone to convict nor prone to acquit is $P_B = 1 - P_A - P_C$.

We can now derive the probabilities of a wrongful conviction, a wrongful acquittal, and a hung jury in a single trial.

A wrongful conviction occurs when (a) the defendant is innocent, which occurs with probability 1 - P(G); (b) the jury is likely to convict (or, equivalently, mN jurors receive the strong signal), which occurs with probability $P_C$; and (c) the evidence is sufficiently strong to meet the reasonable-doubt standard, that is, $s \geq sI - x$. Putting this all together, the probability of a wrongful conviction $P_{WC}$ is given as

$$P_{WC} = [1 - P(G)][1 - F(sI - x|I)]P_C.$$  \hspace{1cm} (1)

16 We follow Neilson and Winter [2000, 2005] in assuming that the probability distribution f of s given the state of the world G or I does not depend on the probability that a juror receives a strong or weak signal [\pi]. Exploring different systems of signals could represent an interesting extension to our jury model. We thank an anonymous referee for this suggestion.

17 For the purpose of our analysis, we assume that a jury reaches a decision by taking a simultaneous vote; i.e., jurors ignore any group strategy aspects and decide independently from other jurors. This means that jurors do not vote against their signal: if a juror receives a guilty [innocent] signal, he votes to convict [acquit]. This assumption, which is the behavior assumed by Condorcet, allows us to isolate the role of our two institutional variables from the possible effects of signaling and informational cascades [e.g., Luppi and Parisi 2013] and the possibility of strategic voting of jurors [e.g., Ladha 1992, Feddersen and Pesendorfer 1998; Kaniovski and Zaigraev 2011].

18 For the purpose of the present analysis, we focus on the outcome of a single trial. Our basic framework can be extended to consider appeals and retrials. See Neilson and Winter (2005).
A wrongful acquittal occurs when (a) the defendant is guilty, which occurs with probability \( P(G) \), but (b) the evidence is not strong enough for a conviction. Formally, the probability of a wrongful acquittal \( P_{WA} \) is given as

\[
P_{WA} = P(G)[P_A s_G + y|G] + (1 - P_A)F[s_G - x|G].
\]

The first term within the squared brackets is the probability that at least \( mN \) jurors receive the weak signal \( (P_A) \), and the evidence is not sufficiently strong to convict \( (s < s_i + y) \). The second term is the probability that at least \( mN \) jurors receive the strong signal \( (P_C) \) or receive both the strong and weak signals \( (1 - P_C - P_A) \), but the evidence is not sufficiently strong to convict \( (s < s_i - x) \).

The probability of a wrongful verdict is given by \( P_W = P_{WC} + P_{WA} \).

A hung jury occurs (a) if the jury is neither prone to convict nor prone to acquit, which happens with probability \( P_B = 1 - P_A - P_C \), and (b) if the true strength of evidence is sufficiently close to the reasonable-doubt standard, that is, it ranges between \( s_i - x \) and \( s_i + y \) (the hung-jury range, as shown in fig. 1). In this range, jurors who receive the strong signal vote to convict, and those who receive the weak signal vote to acquit, resulting in a mistrial. Formally, the probability of a hung jury \( P_H \) is given as

\[
P_H = [1 - P(G)][1 - F(s_G - x|I)]P_B + P(G)[F(s_G + y|G) - F(s_G - x|G)]P_B
\]

where the first term is the probability of a mistrial when the defendant is innocent, and the second term is the probability of a mistrial when the defendant is guilty.

From the equations above, it is straightforward to derive the probability of an accurate verdict; that is, \( P_V = 1 - P_W - P_H \).

The social loss function, which depends on the social costs of a wrongful conviction, of a wrongful acquittal, and of a mistrial, can be expressed as the following:

\[
m \in \min N, m = P_{WC}C_{WC} + P_{WA}C_{WA} + P_HC_H
\]

where \( C_H \), \( C_{WA} \), and \( C_{WC} \) are the monetary social costs for a hung jury, wrongful acquittal, and wrongful conviction, respectively.

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\(^{19}\) As in Neilson and Winter (2000), the administrative costs of increasing \( N \) are omitted. The social function in equation (4) is similar to the social loss function considered by Miceli (1990). The main differences are that Miceli (1990) did not analyze jury size and voting requirement as factors influencing the accuracy of decisions and failed to consider the costs associated with mistrials. Our social function is also comparable to the social loss function considered by Neilson and Winter (2000). The main differences are that Neilson and Winter (2000) did not analyze how different combinations
The objective of the social planner is to minimize the social loss, as expressed in equation (4), by optimally choosing jury size and voting requirement. In section 4, we analyze how different combinations of the two aforementioned institutional variables affect accuracy and the social cost of criminal adjudication through changes in $P_{WCr}$, $P_{WA}$, and $P_H$.

4. OPTIMAL JURY SIZE AND VOTING REQUIREMENT

In this section, we analyze how different combinations of jury size and voting requirement affects trial outcomes.

Let us start by discussing the benchmark case of varying jury size under unanimous verdicts. By restating the Condorcet’s jury theorem under unanimity, we obtain the following lemma:

**Lemma 1 (Jury-Size Effect).** The probability of a wrong verdict decreases in jury size at a decreasing rate. However, given the greater incidence of mistrials, the probability of reaching an accurate unanimous verdict decreases in jury size at a decreasing rate.

**Proof.** See appendix. QED

Lemma 1 unveils an interesting trade-off. Larger juries are less likely to be wrong but are also less likely to reach an accurate verdict because of the greater difficulty in deliberating unanimously.

Next, let us discuss the implications of varying the voting requirement under a given jury size.

**Lemma 2 (Voting Requirement Effect).** For any given jury size, if $\pi = .5$, the probability of a wrong verdict decreases with the voting requirement. However, given the greater incidence of mistrials, the probability of reaching an accurate verdict decreases with the voting requirement. Each probability decreases at an increasing rate for $m \in [.5, \tilde{m}]$ if the voting requirement tends to the majority rule and at a decreasing rate for $m \in [\tilde{m}, 1]$, where $\tilde{m} \in (.5, 1)$.

**Proof.** See appendix. QED

Similar to what we observed in lemma 1, we can see that changes in the voting requirement have a double-edged effect. Relaxing a jury’s voting requirement [i.e., allowing nonunanimous verdicts] facilitates
the reaching of a verdict, but at the same time, it increases the probability of adjudication errors. As the majority requirement is reduced, more verdicts will be reached, but wrongful convictions and wrongful acquittals will also increase.

If each juror has the same probability of receiving a strong signal or weak signal (i.e., $\pi = .5$), lemma 2 reveals the desirability of relaxing the unanimity requirement over relaxing jury size to increase the accuracy of verdicts. Analytically, this result is explained by the fact that $P_v$ is concave with respect to $N$ and convex with respect to $m$ (when $m$ tends to the majority rule). Relaxing the unanimity requirement (under a fixed jury size) generates a “fast” marginal increase in the probability of accurate verdicts, whereas restricting jury size (under unanimity) generates a “slow” marginal increase in the probability of accurate verdicts.

These results can be summarized in the following proposition.

**Proposition 1.** Given lemmas 1 and 2, to increase accuracy of verdicts, relaxing the unanimity requirement down to a majority rule (for any given jury size) is a more effective alternative to restricting jury size (under unanimity) if $\pi = .5$.

**Proof.** See appendix. QED

Under the constraints on jury size and voting requirement set out by the US Supreme Court in *Burch v. Louisiana*, state courts are not allowed to modify jury size and voting requirements simultaneously.

Relaxing the unanimity requirement in favor of a less demanding majority rule yields better results in terms of accuracy of verdicts than a reduction in jury size. Relaxing unanimity increases accuracy at a faster rate than the corresponding change obtainable with a reduction in jury size.

Consistent with *Burch v. Louisiana*, proposition 1 implies that jury size and voting requirements inversely depend on one another. The accuracy of verdicts is maximized when requiring unanimous verdicts for small juries or allowing nonunanimous verdicts with large juries. These findings provide a rationale for the constraints introduced by the US Supreme Court in *Burch v. Louisiana*: a combined use of small juries and nonunanimous verdicts would not be desirable in criminal adjudication.

5. **Conclusion**

Let us now step back to review the previously stated results from a bird’s-eye perspective. Our findings help evaluate the effect of the
changes to jury structure brought about by the US Supreme Court and state legislation. The results on the capacity of a jury to reach an accurate verdict, taken in isolation, provide an economic rationale for the constraints introduced by the *Burch v. Louisiana* decision. Large nonunanimous juries or small unanimous juries are alternative ways to maximize the accuracy of verdicts while preserving the functionality of juries. In the choice between these alternatives, the vast majority of jurisdictions retained the unanimity rule, and there is near universal acceptance to require it for capital murder cases given the severity of the consequences resulting from wrongful convictions. In these cases, the probability of convicting an innocent person should be kept to a minimum, avoiding as much error as possible. Notwithstanding the limited adoption of nonunanimous juries in US state courts, our results lend support to the elimination of the unanimity requirement in the presence of large juries. As we move away from capital murder cases, combining a qualified majority rule with larger juries would seem desirable, inasmuch as the undesirability gap between wrongful convictions and wrongful acquittals narrows. Optimal jury size can even fall below the lower limit of six members set by the US Supreme Court in the *Burch v. Louisiana* case, inasmuch as juries are required to decide unanimously. Our result largely aligns with the conventional wisdom in existing literature. When unanimity is required, the use of smaller juries could reduce the probability of a single juror causing a deadlock and may be desirable to empower the jury with the capacity to reach a verdict.

Future research in this field should extend our analysis to investigate how optimal jury design would change when considering retrials (Neilson and Winter 2005), correlated votes (Rubinfeld and Sappington 1987), endogenous social values of adjudication errors (Miceli 1990), behavioral cascades (Luppi and Parisi 2013), and strategic voting by jurors (Feddersen and Pesendorfer 1998). For all these extensions, our model could usefully serve as a building block for the understanding of more complex jury decision-making scenarios. Finally, as shown in Pi, Parisi, and Luppi (2020), the choice of different Blackstonian ratios by US jurisdictions indirectly implies the jurisdiction’s commitment to different “beyond a reasonable doubt” thresholds. Our next research objective is to explore how the jurisdictions’ choices of different standards of proof influence their choices regarding jury size and voting requirements (Guerra and Parisi 2019).
APPENDIX

Proofs

Proof of Lemma 1. Given \( m = 1 \), \( \partial P_{WC}/\partial N = [1 - P(G)][1 - F(s_i - x[I])] \pi^N \ln \pi \) and \( \partial P_{WA}/\partial N = [F(s_i + y|G) - F(s_i - x|G)](1 - \pi)^N \ln(1 - \pi) \), which are both negative because \( \pi \in [0, 1] \). Hence, \( \partial P_W/\partial N < 0 \). The second-order derivatives are equal to \( [1 - P(G)][1 - F(s_i - x[I])] \pi^N \ln^2 \pi \) and \( [F(s_i + y|G) - F(s_i - x|G)](1 - \pi)^N \ln^3(1 - \pi) \), respectively, which are both positive. Hence, \( \partial^2 P_w/\partial N^2 > 0 \).

Next, we prove that \( \partial P_H/\partial N > 0 \). Given \( m = 1 \), \( \partial P_H/\partial N = \partial P_B/\partial N \{[1 - P(G)][1 - F(s_i - x[I]) + P(G)[F(s_i + y|G) - F(s_i - x|G)]\} \). Because \( \partial P_B/\partial N = \{[1 - \pi]^N \ln(1 - \pi) + \pi^N \ln \pi \} \), which is positive as \( \pi \in [0, 1] \), it follows that \( \partial P_H/\partial N > 0 \).

Finally, we prove that \( \partial P_V/\partial N < 0 \), \( \partial^2 P_V/\partial N^2 > 0 \). Given \( m = 1 \), \( \partial P_V/\partial N = [1 - P(G)][1 - F(s_i - x[I])][1 - \pi)^N \ln(1 - \pi) + P(G)[F(s_i + y|G) - F(s_i - x|G)] \pi^N \ln \pi \), which is negative because \( \pi \in [0, 1] \). The second-order derivative is equal to \( [1 - P(G)][1 - F(s_i - x[I])][1 - \pi)^N \ln^3(1 - \pi) + P(G)[F(s_i + y|G) - F(s_i - x|G)] \pi^N \ln^2 \pi \), which is positive. QED

Proof of Lemma 2. For a given \( N \), \( \partial P_{WC}/\partial m = [1 - P(G)][1 - F(s_i - x[I])] \partial P_C/\partial m \), where \( \partial P_C/\partial m \) is equal to

\[
\pi^m N(1 - \pi)^m N \left\{ \frac{N!}{mN!(N - m)!} N \ln \left( \frac{\pi}{1 - \pi} \right) + \frac{\partial}{\partial m} \left( \frac{N!}{mN!(N - m)!} \right) \right\},
\]

which is negative if \( \pi \leq .5 \). For a given \( N \), \( \partial P_{WA}/\partial m = P(G)[F(s_i + y|G) - F(s_i - x|G)] \partial P_A/\partial m \), where \( \partial P_A/\partial m \) is equal to

\[
(1 - \pi)^m N^m \left\{ \frac{N!}{mN!(N - m)!} N \ln \left( \frac{\pi}{1 - \pi} \right) + \frac{\partial}{\partial m} \left( \frac{N!}{mN!(N - m)!} \right) \right\},
\]

which is negative if \( \pi \geq .5 \). Hence, \( \partial P_w/\partial m < 0 \) if \( \pi = .5 \).

Next, we prove that \( \partial P_H/\partial m > 0 \). For a given \( N \), because \( \partial P_B/\partial m = \{-[\partial P_C/\partial m + \partial P_A/\partial m], \) whereby \( \partial P_C/\partial m < 0 \) if \( \pi \leq .5 \) and \( \partial P_A/\partial m < 0 \) if \( \pi \geq .5 \), it follows that \( \partial P_H/\partial m > 0 \) if \( \pi = .5 \).

Finally, we prove that \( \partial P_V/\partial m < 0 \). For a given \( N \), \( \partial P_V/\partial m = [1 - P(G)][1 - F(s_i - x[I])] \partial P_A/\partial m + P(G)[F(s_i + y|G) - F(s_i - x|G)] \partial P_C/\partial m \), which is negative if \( \pi = .5 \).

Finally, let us analyze second-order partial derivatives with respect to \( m \). If \( \pi = .5 \), the second-order partial derivative of \( P_{WC} \) with respect to \( m \) is

\[
[1 - P(G)][1 - F(s_i - x[I])].5 \frac{N!}{mN!(N - m)!} \frac{\partial^2}{\partial m^2} \left( \frac{N!}{mN!(N - m)!} \right).
\]
To compute the second-order derivative of the factorial function, let us consider $N = 6$ and marginal increases of .1 in $m$ from $m = .5$ to $m = 1$. The computations show that the factorial function is decreasing in $m$ at an increasing rate for $m \in [.5, .8]$ and at a decreasing rate for $m \in [.8, 1]$. A similar trend occurs if $N = 12$, whereby the factorial function is decreasing in $m$ at an increasing rate for $m \in [.5, .7]$ and at a decreasing rate for $m \in [.7, 1]$. Let $\tilde{m}$ denote the inflection point. Thus, $\partial^2 P_{WC}/\partial m^2 < 0$ for $m \in [.5, \tilde{m}]$, and $\partial^2 P_{WC}/\partial m^2 > 0$ for $m \in (\tilde{m}, 1)$. The same holds for $\partial^2 P_{WA}/\partial m^2$, which is negative. Hence, $\partial^2 P_{W}/\partial m^2 < 0$. QED

**Proof of Proposition 1.** This follows from proofs of lemmas 1 and 2, because $\partial^2 P_{V}/\partial N^2 > 0$, whereas $\partial^2 P_{V}/\partial m^2 < 0$ if $\pi = .5$ and $m \in [0.5, m]$. QED

**REFERENCES**


